

The strong mixing and the operator-selfdecomposability properties.

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February 27, 2014

Abstract. For nonstationary, strongly mixing sequences of random variables taking their values in a finite-dimensional Euclidean space, with the partial sums being normalized via matrix multiplication, with certain standard conditions being met, the possible limit distributions are precisely the operator-selfdecomposable laws.

When one has observations (distributions) with values in an algebraic structure then their normalizations should be consistent with the structure in question. Thus, when ξ_1, ξ_2, \dots are \mathbb{R}^d -valued vectors then one should consider sums

$$A_n(\xi_1 + \xi_2 + \dots + \xi_n) + x_n, \quad (*)$$

where (A_n) are linear operators (matrices) on \mathbb{R}^d . Similarly, if the algebraic structure is a Banach space or a topological group then the (A_n) 's in $(*)$ should be bounded linear operators or automorphisms of the group, respectively. That novel paradigm required completely new algebraic methods and tools such as decomposability semigroups associated with probability measures, the Numakura Theorem on idempotents in (abstract) topological semigroups or elements of Lie theory. Sharpe (1969), for independent identically distributed ξ'_i s in $(*)$, and Urbanik (1972) and (1978), for infinitesimal triangular arrays $(A_n \xi_i, 1 \leq i \leq n, n \geq 1)$, described limit distributions in the scheme $(*)$. The monograph Jurek and Mason (1993) summarized the research in that area for stochastically independent variables. However, the

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[‡]Research funded by Narodowe Centrum Nauki (NCN) grant no Dec2011/01/B/ST1/01257.

CLT for affine normalizations from Hahn and Klass (1981) still awaits for a coordinate-free proof.

Here we will describe limiting distributions of $(*)$ for \mathbb{R}^d -valued random variables ξ_1, ξ_2, \dots , that are only strongly mixing, as defined by Rosenblatt (1956). Classical limiting distributions for strongly mixing sequences normalized by scalars are described in the monograph by Bradley (2007).

1. Strong mixing and operator-selfdecomposability.

Let \mathbb{R}^d be the d -dimensional Euclidean space. As in Jurek and Mason (1993), by $\mathbf{End}(\mathbb{R}^d) \equiv \mathbf{End}$ we denote the Banach algebra of all bounded linear operators (matrices) on \mathbb{R}^d and by $\mathbf{Aut}(\mathbb{R}^d) \equiv \mathbf{Aut}$ the group of all linear bounded and invertible operators (matrices). By $\mathcal{P}(\mathbb{R}^d) \equiv \mathcal{P}$ we denote the topological semigroup of all Borel probability measures on \mathbb{R}^d with convolution $*$ and weak convergence topology.

Furthermore, let (Ω, \mathbb{F}, P) be a probability space rich enough to carry uncountable family of independent uniformly distributed random variables as well as sequences $\mathbf{X} := (X_1, X_2, \dots)$ of \mathbb{R}^d -valued random vectors (in short: random vectors); cf. Dudley (2002), Theorem 8.2.2.

We will say that a random vector X or its probability distribution μ is *full or genuinely d dimensional* if its support is not contained in any proper hyperplane in \mathbb{R}^d . (Recall that a *hyperplane* is a linear subspace of \mathbb{R}^d shifted by a vector.) By \mathcal{F} we denote the family of all full measures. It is an open (in weak convergence topology) subsemigroup of \mathcal{P} cf. Jurek-Mason (1993), Corollary 2.1.2.

With a random vector X or its probability distribution μ we associate two semigroups of matrices: the *Urbanik decomposability semigroup* $\mathbf{D}(X)$ (or $\mathbf{D}(\mu)$) and the *symmetry semigroup* $\mathbf{A}(X)$ (or $\mathbf{A}(\mu)$) as follows:

$$\begin{aligned} \mathbf{D}(X) &:= \{A \in \mathbf{End} : X \stackrel{d}{=} AX + Y \text{ for some } Y \text{ independent of } X\}, \\ \mathbf{A}(X) &:= \{A \in \mathbf{End} : X \stackrel{d}{=} AX + a \text{ for some vector } a \in \mathbb{R}^d\}, \end{aligned} \quad (1)$$

where $\stackrel{d}{=}$ denotes the equality in distribution. In an analogous way we define semigroups $\mathbf{D}(\mu)$ and $\mathbf{A}(\mu)$. Of course, $\mathbf{A}(X) \subset \mathbf{D}(X)$ and the operators 0 (zero) and I (identity) are always in $\mathbf{D}(X)$.

(The symbol 0 will be used freely for the zero elements of \mathbb{R} , \mathbb{R}^d , and \mathbf{End} . In context, that should not cause confusion.)

For the references below let us recall that

- (i) $\mathbf{D}(\mu)$ is a compact semigroup in $\mathbf{End}(\mathbb{R}^d)$ iff μ is a full measure
iff $\mathbf{A}(\mu)$ is a compact group in $\mathbf{Aut}(\mathbb{R}^d)$;
- (ii) If μ is full, then $\mathbf{A}(\mu)$ is the largest group in the Urbanik
semigroup $\mathbf{D}(\mu)$. (2)

Cf. Jurek and Mason (1993), Theorem 2.3.1 and Corollary 2.3.2 and Proposition 2.3.4.

We will say that a probability measure μ is operator-selfdecomposable if there exist a sequence $b_n \in \mathbb{R}^d$, a sequence $A_n \in \mathbf{End}$ and a sequence X_n of independent \mathbb{R}^d -valued random vectors such that

- (i) the triangular array $(A_n X_j : 1 \leq j \leq n, n \geq 1)$ is infinitesimal;
- (ii) $\lim_{n \rightarrow \infty} A_n(X_1 + X_2 + \dots + X_n) + b_n \Rightarrow \mu$.

(Condition (i) simply means that $A_n X_j \rightarrow 0$ in probability as $n \rightarrow \infty$, $j \in \{1, \dots, n\}$.)

The main characterization due to K. Urbanik is as follows:

A full measure μ is operator-selfdecomposable iff its decomposability semigroup $\mathbf{D}(\mu)$ contains at least one one-parameter semigroup $\{\exp(-tQ), t \geq 0\}$ with $\exp(-tQ) \rightarrow 0$ (the zero matrix) as $t \rightarrow \infty$.

Cf. Jurek-Mason (1993), Theorem 3.3.5. (The stipulation in that theorem that Q be invertible, is superfluous; note that $Q^{-1} = \int_0^\infty e^{-sQ} ds$; the integral is well defined because $\exp(-tQ) \rightarrow 0$ as $t \rightarrow \infty$.)

Also it might be of some importance to mention here that we have the following random integral representation:

$$\mu \text{ is operator-selfdecomposable iff } \mu = \mathcal{L}\left(\int_0^\infty e^{-tQ} dY(t),\right)$$

for some Lévy process Y (so called *background driving Lévy process* (BDLP); cf. Jurek (1982) or Jurek-Mason (1993), Theorem 3.6.6.

Since our aim here is to extend the notion of operator-selfdecomposability to some dependent random variables, let's recall that for two sub- σ -fields \mathcal{A} and \mathcal{B} of \mathbb{F} we define the measure of dependence α between them as follows:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

For a given sequence $\mathbf{X} := (X_1, X_2, \dots)$ of \mathbb{R}^d -valued random variables, we define for each positive integer n the dependence coefficient

$$\alpha(n) \equiv \alpha(\mathbf{X}; n) := \sup_{j \in \mathbf{N}} \alpha(\sigma(X_k, 1 \leq k \leq j), \sigma(X_k, k \geq j+n)), \quad (3)$$

where $\sigma(\dots)$ denotes the σ -field generated by (\dots) . We will say that a sequence \mathbf{X} is *strongly mixing* (Rosenblatt (1956)) if

$$\alpha(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4)$$

Of course, if the elements of \mathbf{X} are stochastically independent then $\alpha(\mathbf{X}; n) \equiv 0$.

THEOREM 1. *Let $\mathbf{X} := (X_1, X_2, \dots)$ be sequence \mathbb{R}^d -valued random vectors with the partial sums $S_n := X_1 + X_2 + \dots + X_n$, and let $(A_n) \in \mathbf{End}(\mathbb{R}^d)$ and $(b_n) \in \mathbb{R}^d$ be sequences of bounded linear operators and vectors, respectively satisfying conditions:*

- (i) $\alpha(\mathbf{X}; n) \rightarrow 0$ as $n \rightarrow \infty$, i.e., the sequence \mathbf{X} is strongly mixing;
- (ii) the triangular array $(A_n X_j, 1 \leq j \leq n, n \geq 1)$ is infinitesimal;
- (iii) $A_n S_n + b_n \Rightarrow \mu$, for some full probability measure μ .

Then the limit distribution μ is operator-selfdecomposable, that is, there exists a one parameter semigroup $\{e^{-tQ} : t \geq 0\} \subset \mathbf{D}(\mu)$ with $\lim_{t \rightarrow \infty} e^{-tQ} = 0$.

The line of reasoning in our proof of this theorem is as follows. First, in Section 2 we investigate the normalizing sequence (A_n) of matrices, and in particular we show that one may choose a more appropriate sequence (\tilde{A}_n) . Then in Section 3, using the new normalizing sequence, we construct in a few steps a one-parameter semigroup $(e^{tQ}, t \geq 0)$. Here, we follow Urbanik (1972); but we could also argue similarly as in Urbanik (1978) or Jurek-Mason (1993), where the proof is valid in infinite dimensional linear spaces.

2. Auxiliary propositions and lemmas.

First, some consequences of the operator-convergence of types theorems (Section 2.2 in Jurek-Mason (1993)):

PROPOSITION 1. *Under the assumptions (ii) and (iii) in Theorem 1,*

- a) $A_n \rightarrow 0$ as $n \rightarrow \infty$; the inverse A_n^{-1} exists for all sufficiently large n ;
- b) there exist \tilde{A}_n for which (ii) and (iii) (in Theorem 1) hold and $\tilde{A}_{n+1} \tilde{A}_n^{-1} \rightarrow I$ (the identity matrix);
- c) one has that

$$\lim_{n \rightarrow \infty} \left| \frac{\det A_{n+1}}{\det A_n} \right| = \lim_{n \rightarrow \infty} \frac{\det \tilde{A}_{n+1}}{\det \tilde{A}_n} = 1.$$

Cf. Jurek-Mason (1993), Section 3.2 : Propositions 3.2.1 and 3.2.2. For part c) one needs also Corollaries 2.3.2 and 2.4.2 as $\tilde{A}_n := H_n A_n$ for some $H_n \in \mathbf{A}(\mu)$.

Second, a note on uniform infinitesimal triangular arrays.

LEMMA 1. *Suppose that for each $n \in \mathbb{N}$, I_n is a nonempty set. Suppose that for each $n \in \mathbb{N}$ and each $j \in I_n$, $X_{n,j}$ is Banach space valued random element. The the following two statements are equivalent:*

$$(A) \quad \forall(\epsilon > 0) \quad \lim_{n \rightarrow \infty} \sup_{j \in I_n} P(\|X_{n,j}\| \geq \epsilon) = 0 \quad (5)$$

There exists a sequence $\delta_1 \geq \delta_2 \geq \dots \geq \delta_{n-1} \geq \delta_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$(B) \quad \forall(n \in \mathbb{N}) \quad \forall(j \in I_n) \quad P(\|X_{n,j}\| \geq \delta_n) \leq \delta_n \quad (6)$$

Proof. $((A) \Rightarrow (B))$ Let $N_1 := 1$. For already defined N_1, N_2, \dots, N_{m-1} , let $N_m > N_{m-1}$ be such that

$$\forall(n \geq N_m) \quad \forall(j \in I_n) \quad P(\|X_{n,j}\| \geq 1/m) \leq \frac{1}{m} \quad (7)$$

which always exists by (5). Next, for the defined sequence

$$1 = N_1 < N_2 < \dots < N_n < \dots,$$

let us define the sequence (δ_n) as follows: For each $m \in \mathbb{N}$,

$$\delta_n := 1/m, \quad \text{for all } n \text{ such that } N_m \leq n \leq N_{m+1} - 1. \quad (8)$$

Thus by virtue of the above construction, $\delta_n \rightarrow 0$ as $n \rightarrow \infty$; and for each $n \in \mathbb{N}$ there exists exactly one $m \in \mathbb{N}$ such that $N_m \leq n \leq N_{m+1} - 1$, and by (7) and (8),

$$P(\|X_{n,j}\| \geq \delta_n) = P(\|X_{n,j}\| \geq 1/m) \leq 1/m = \delta_n \quad \text{for all } j \in I_n,$$

which completes the proof $(A) \Rightarrow (B)$. The implication $(B) \Rightarrow (A)$ is obvious.

COROLLARY 1. *For the infinitesimal triangular array $(X_{n,j})$ as in Lemma 1 and $q_n \rightarrow \infty$, $q_n \leq \delta_n^{-1/2}$ then for any set $Q \subset I_n$ such that $\text{card} Q \leq q_n$ we have that*

$$P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n^{1/2}) \leq \delta_n^{1/2}$$

Since $\text{card } Q \leq q_n$ and $\delta_n q_n \leq \delta_n^{1/2}$

$$\begin{aligned} P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n^{1/2}) &\leq P(\|\sum_{k \in Q} X_{n,k}\| \geq \delta_n q_n) \\ &\leq \sum_{k \in Q} P(\|X_{n,k}\| \geq \delta_n) \leq q_n \delta_n \leq \delta_n^{1/2}. \end{aligned}$$

Third, a generalization of Proposition 3.2.3 in Jurek-Mason, for strongly mixing sequences.

PROPOSITION 2. *Suppose the hypothesis of Theorem 1, including all of conditions (i), (ii), and (iii) there, hold. Suppose also that for every $n \in \mathbb{N}$, the matrix A_n is invertible. Then*

$$\sup\{\|A_n A_m^{-1}\| : n \in \mathbb{N}, 1 \leq m \leq n\} < \infty. \quad (9)$$

Moreover, if for each $n \in \mathbb{N}$, m_n is an integer such that $1 \leq m_n \leq n$, then all limits points of the sequence $(A_n A_{m_n}^{-1})_{n \in \mathbb{N}}$ are in $D(\mu)$.

Proof. We shall first prove (9). Suppose that for each $n \in \mathbb{N}$, m_n is an integer such that $1 \leq m_n \leq n$. To prove (9), it suffices to prove that

$$\sup_{n \in \mathbb{N}} \|A_n A_{m_n}^{-1}\| < \infty. \quad (10)$$

If instead $\|A_n A_{m_n}^{-1}\| \rightarrow \infty$ along some subsequence of $n \in \mathbb{N}$, then within that subsequence the integers m_n could not be bounded (for otherwise $\|A_n A_{m_n}^{-1}\| \rightarrow 0$ would occur along that subsequence by Proposition 1(a)), and there would be a further subsequence along which $m_n \rightarrow \infty$. Letting $m_n := n$ for all n not in that “further subsequence,” we have reduced our task (for the proof of (9)) to proving (10) under the additional assumption that $m_n \rightarrow \infty$ as $n \rightarrow \infty$.

For the rest of the proof it is assumed that

$$m_n \rightarrow \infty, \quad 0 < q_n \rightarrow \infty \quad \text{and} \quad q_n \leq \delta_n^{-1/2},$$

where the sequence $(\delta_n)_{n \in \mathbb{N}}$ is as in Lemma 1.

For $n \in \mathbb{N}$ define random vectors as follows:

$$\eta_n := \begin{cases} 0, & \text{if } m_n = n; \\ S(\mathbf{X}, n) - S(\mathbf{X}, m_n), & \text{if } n - q_n \leq m_n < n \\ S(\mathbf{X}, m_n + q_n) - S(\mathbf{X}, m_n), & \text{if } m_n \leq n - q_n - 1. \end{cases}$$

and

$$\xi_n := \begin{cases} 0, & \text{if } n - q_n \leq m_n \leq n \\ S(\mathbf{X}, n) - S(\mathbf{X}, m_n + q_n), & \text{if } m_n \leq n - q_n - 1. \end{cases}$$

Thus

$$S_n \equiv S(\mathbf{X}, n) = S(\mathbf{X}, m_n) + \eta_n + \xi_n \quad (11)$$

Since η_n is either zero or the sum of at most $q_n \leq \delta_n^{-1/2}$ of the variables X_k , one has by Corollary 1 that $A_n \eta_n \rightarrow 0$ in probability as $n \rightarrow \infty$. Consequently,

$$A_n S(\mathbf{X}, m_n) + A_n \xi_n + b_n \Rightarrow Z, \quad \text{as } n \rightarrow \infty. \quad (12)$$

>From the description of ξ_n 's, for the case $m_n \leq n - q_n - 1$ we have

$$\alpha(\sigma(S(\mathbf{X}, m_n)), \sigma(\xi_n)) \leq \alpha(\mathbf{X}, q_n + 1) \quad (13)$$

In the opposite case ($m_n \geq n - q_n$), the ξ_n 's are zero (constant variables) and the left-hand side of (13) is therefore zero. Now from (12),

$$[A_n A_{m_n}^{-1} (A_{m_n} S(\mathbf{X}, m_n) + b_{m_n})] + [A_n \xi_n + b_n - A_n A_{m_n}^{-1} b_{m_n}] \Rightarrow Z, n \rightarrow \infty \quad (14)$$

For simplicity, let V_n and W_n denote the first and the second expressions in the above square brackets, that is

$$V_n + W_n \Rightarrow Z \quad (15)$$

>From (13) and Corollary 1.11 in Bradley (2007), Vol. 1,

$$\begin{aligned} |\mathbb{E}[\exp i < t, V_n + W_n >] - \mathbb{E}[\exp i < t, V_n >] \cdot \mathbb{E}[\exp i < t, W_n >]| \\ \leq 16 \alpha(\mathbf{X}, q_n + 1) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence by (15),

$$\mathbb{E}[\exp i < t, V_n >] \cdot \mathbb{E}[\exp i < t, W_n >] \rightarrow \mathbb{E}[\exp i < t, Z >] \quad \text{as } n \rightarrow \infty.$$

Our next task is to replace vectors W_n by vectors that are stochastically independent of V_n . To this aim, let $\zeta_{n,m}$, $m = 1, 2, \dots, n$ be random vectors independent of $\sigma(\mathbf{X}, Z)$ such that

$$\begin{aligned} \zeta_{n,m} &:= b_n - A_n A_m^{-1} b_m, \quad \text{if } m + q_n \geq n; \text{ (a constant); for } m + q_n < n, \\ \mathcal{L}(\zeta_{n,m}) &:= \mathcal{L}(A_n (S(\mathbf{X}, n) - S(\mathbf{X}, m + q_n)) + b_n - A_n A_m^{-1} b_m). \end{aligned}$$

But note that $\mathcal{L}(W_n) = \mathcal{L}(\zeta_{n,m_n})$ and thus by the above,

$$\mathbb{E}[\exp i < t, V_n >] \cdot \mathbb{E}[\exp i < t, \zeta_{n,m_n} >] \rightarrow \mathbb{E}[\exp i < t, Z >] \quad \text{as } n \rightarrow \infty$$

and therefore

$$\mathcal{L}(V_n) * \mathcal{L}(\zeta_{n,m_n}) = \mathcal{L}(V_n + \zeta_{n,m_n}) \Rightarrow Z$$

By Parthasarathy (1967), Theorem 2.2 in Chapter III, $(\mathcal{L}(V_n))_n$ is shift compact, or in the “symmetrization” terminology there,

$$(A_n A_{m_n}^{-1} (A_{m_n} S(\mathbf{X}^\circ, m_n)))_{n \in \mathbb{N}}, \text{ is compact and } A_{m_n} S(\mathbf{X}^\circ, m_n) \Rightarrow Z^\circ.$$

>From Lemma 2.2.3 in Jurek-Mason (1993) we now get the boundedness in Proposition 2.

Further, if D is a limit point of the family of matrices $(A_n A_{m_n}^{-1})_{n \in \mathbb{N}}$ then from (14) we get $\mathcal{L}(DZ + Y) = \mathcal{L}(Z)$ for some random variable Y , a limit point of $(\zeta_{n,m_n})_{n \in \mathbb{N}}$, independent of Z . This completes the proof Proposition 2.

3. Construction of the one-parameter semigroup in $\mathbf{D}(\mu)$.

Here we follow the Urbanik construction from Urbanik (1972); see also Jurek-Mason (1993), Section 3.3. Throughout this section, as in the hypothesis of Theorem 1, we assume that *the probability measure μ is full*; and as allowed by Proposition 1, we assume that the matrices A_n satisfying conditions (ii) and (iii) in Theorem 1 are invertible and satisfy $A_{n+1} A_n^{-1} \rightarrow I$ and (hence) $\det A_{n+1} / \det A_n \rightarrow 1$ (as $n \rightarrow \infty$).

By an *idempotent* J , in $\mathbf{End}(\mathbb{R}^d)$, we mean a projector from \mathbb{R}^d onto the linear subspace $J(\mathbb{R}^d)$, that is $J^2 = J$. Following Numakura (1952), p. 103, we will say that *idempotent K is under idempotent J* , if $K \neq J$ and $JK = KJ = K$. Hence, in particular, $K(\mathbb{R}^d) \subsetneq J(\mathbb{R}^d)$. If there is no non-zero idempotent under J , then we will say that J is a *primitive idempotent*.

Idempotents will play a crucial role below as we have the following: *for an idempotent J we have that*

$$J \in \mathbf{D}(\mu) \quad \text{iff} \quad (I - J) \in \mathbf{D}(\mu) \quad \text{and} \quad \mu = J\mu * (I - J)\mu$$

Furthermore, if an idempotent K is under J and both are in $\mathbf{D}(\mu)$ then

$$\mu = K\mu * (J - K)\mu * (I - J)\mu \quad \text{and} \quad K + (J - K) + (I - J) = I \quad (16)$$

for details cf. Jurek-Mason (1993), Theorem 2.3.6.

Below $\det_J A$ means the determinant of matrix representation of the operator JA in $J(\mathbb{R}^d)$ relatively to an orthogonal basis of $J(\mathbb{R}^d)$. Hence we get

$$\begin{aligned} (a) \quad & \det_J A = \det_J(JA) = \det_J(AJ) = \det_J(JAJ); \\ (b) \quad & \det_J(AJB) = \det_J A \det_J B; \\ (c) \quad & \det(JAJ + (I - J)B(I - J)) = \det_J A \det_{I-J} B. \end{aligned} \quad (17)$$

LEMMA 2. For a given idempotent $J \in \mathbf{D}(\mu)$, for each $0 < c < 1$ there exist $K_c \in \mathbf{D}(\mu)$ such that $\det_J K_c = c$

Proof. For $1 \leq n \leq m$, one has the inequalities

$$\begin{aligned} \|A_{m+1}A_n^{-1} - A_mA_n^{-1}\| &\leq \|A_mA_n^{-1}\| \|A_{m+1}A_m^{-1} - I\| \\ &\leq (\sup_{n \leq m} \|A_mA_n^{-1}\|) \|A_{m+1}A_m^{-1} - I\|. \end{aligned}$$

Since by Proposition 2, $\sup\{\|A_mA_m^{-1}\| : 1 \leq m \leq n, n \in \mathbb{N}\} < \infty$, we get

$$\lim_{m \rightarrow \infty} \sup_{n \leq m} \|\|A_{m+1}A_n^{-1}\| - \|A_mA_n^{-1}\|\| \leq \lim_{m \rightarrow \infty} \sup_{n \leq m} \|A_{m+1}A_m^{-1} - I\| = 0.$$

Since the functions $\mathbb{R}^{d^2} \ni A \rightarrow \|A\|$ and $\mathbb{R}^{d^2} \ni A \rightarrow \det_J A$ are continuous therefore by putting $b_{m,n} := \det_J A_mA_n^{-1}$ ($n \leq m$) we infer that

$$b_{n,n} = 1, \quad \lim_{m \rightarrow \infty} b_{m,n} = 0 \quad (n = 1, 2, \dots), \quad \lim_{m \rightarrow \infty} \sup_{n \leq m} |b_{m+1,n} - b_{m,n}| = 0. \quad (18)$$

Thus for any $0 < c < 1$ and the sequence $m_n := \sup\{k \geq n : b_{k,n} \geq c\}$ we get $b_{m_n+1,n} < c \leq b_{m_n,n}$, so from (18), $\lim_{n \rightarrow \infty} b_{m_n,n} = c$.

Furthermore, by Proposition 2, if K_c is a limit point of a sequence $(A_{m_n}A_n^{-1})$ then K_c is in $\mathbf{D}(\mu)$ and, by (18), $\det_J K_c = c$, which concludes the proof.

LEMMA 3. Let J be non-zero idempotent in $\mathbf{D}(\mu)$. Then there exists $T_n \in \mathbf{D}(\mu)$, $n = 1, 2, \dots$ such that

$$JT_n = T_nJ = T_n, \quad T_n \rightarrow J \text{ and } \lim_{k \rightarrow \infty} T_n^k = 0 \quad (n = 1, 2, \dots) \quad (19)$$

Proof. We shall justify the above claim by the mathematical induction with respect to the dimension of linear space $J(\mathbb{R}^d)$.

Step 1. $\dim J(\mathbb{R}^d) = 1$.

>From Lemma 2, there exist $K_n \in \mathbf{D}(\mu)$ such that $\det_J K_n = 1 - 1/n$. Putting $T_n := JK_nJ$ we have that the linear transformation $T_n : J(\mathbb{R}^d) \rightarrow J(\mathbb{R}^d)$ must be a multiple of J ; ($\dim J(\mathbb{R}^d) = 1$). But $\det_J T_n = \det_J K_n = 1 - 1/n$ and thus $T_n = (1 - 1/n)J$ which satisfies (19).

Step 2. Assume $\dim J(\mathbb{R}^d) = l > 1$ and for all idempotents $K \in \mathbf{D}(\mu)$ such that $\dim K(\mathbb{R}^d) < l$, Lemma 3 is true.

Case (i). Assume that there exist non-zero idempotent $L \in \mathbf{D}(\mu)$ such that $L \neq J$ and

$$L = JL = LJ, \quad (20)$$

that is, the idempotent J is not a primitive one.

From the above $J - L$ is also an idempotent. From Jurek-Mason (1993) Theorem 2.3.6 (a), $I - L \in \mathbf{D}(\mu)$. Hence $J(I - L) = J - L \in \mathbf{D}(\mu)$.

Since $\dim L(\mathbb{R}^d) < l$ and $\dim(J - L)(\mathbb{R}^d) < l$ therefore, by the mathematical induction assumption, there exist sequences (U_n) and (V_n) in $\mathbf{D}(\mu)$ such that

$$U_n \rightarrow L, \quad LU_n = U_n L = U_n \quad \text{and} \quad \lim_{k \rightarrow \infty} U_n^k = 0 \quad (n = 1, 2, \dots)$$

$$V_n \rightarrow J - L, \quad (J - L)V_n = V_n(J - L) = V_n \quad \text{and} \quad \lim_{k \rightarrow \infty} V_n^k = 0 \quad (n = 1, 2, \dots)$$

Then putting $T_n := U_n + V_n$ we have $T_n \rightarrow J$. Further, from the identity $T_n = LU_n L + (I - L)V_n(I - L)$ and again by Theorem 2.3.6 (d) in Jurek-Mason (1993) we get that $T_n \in \mathbf{D}(\mu)$ and also $T_n^k = U_n^k + V_n^k \rightarrow 0$ as $k \rightarrow \infty$. (Also, the first two equalities in (19) hold by an elementary argument.) This completes the Case (i).

Case(ii). There are no non-zero idempotents L in $\mathbf{D}(\mu)$ different from J and satisfying $JL = LJ = L$, i.e., idempotent J is a primitive idempotent.

>From Lemma 2, choose $D_n \in \mathbf{D}(\mu)$ such that

$$0 < \det_J D_n < 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \det_J D_n = 1. \quad (21)$$

By (a) in the formula (17) we may assume that $JD_n = D_n J = D_n$ and if D is a limit point of the sequence D_n then we also have equalities

$$D \in \mathbf{D}(\mu), \quad JD = DJ = D \quad \text{and} \quad \det_J D = 1 \quad (22)$$

Put $A := D + I - J$. Note that $A = JDJ + (I - J)I(I - J)$. Then by (d) in Theorem 2.3.6 from Jurek-Mason (1993) $A \in \mathbf{D}(\mu)$. However, by (17) and (21),

$$\det A = \det_{J+I-J}(JDJ + (I - J)I(I - J)) = \det_J D \det_{I-J}(I - J) = 1.$$

Consequently, by Jurek-Mason (1993), Proposition 2.3.5 and Corollary 2.3.2,

$$A \in \mathbf{A}(\mu) \text{ (a compact group in } \mathbf{Aut} \text{) and } A^{r_n} \rightarrow I, \quad (23)$$

for some $r_1 < r_2 < \dots$. Since $JA^n = D^n$ we have that $D^{r_n} \rightarrow J$. Furthermore, since D is a limit point of the sequence D_n we can choose a subsequence (k_n) such that

$$T_n := D_{k_n}^{r_n} \rightarrow J; \quad JT_n = T_n J = T_n; \quad 0 < \det_J T_n < 1.$$

To complete the proof one needs to show that $T_n^k \rightarrow 0$ as $k \rightarrow \infty$.

For each n , the monothetic semigroup $\text{sem}(T_n)$ (the smallest closed sub-semigroup containing T_n) is compact in $\mathbf{D}(\mu)$. By the Numakura Theorem (Corollary 1.1.3 in Jurek-Mason) the limit points of $(T_n^k)_{k \in \mathbb{N}}$ form a group, denoted by $K(T_n)$, with the unit L that satisfies

$$JL = LJ = L \quad \text{and} \quad \det_J L = 0 \quad \text{and thus} \quad L \neq J$$

Because of the assumption (ii) we must have $L = 0$. Consequently $T_n^k \rightarrow 0$ as $k \rightarrow \infty$, which completes the proof of Lemma 3.

Using the formula (16) inductively, there are finitely many non-zero primitive idempotents J_1, J_2, \dots, J_q in $\mathbf{D}(\mu)$, $q \leq d$ (the dimension of \mathbb{R}^d), such that

$$I = J_1 + J_2 + \dots + J_q, \quad J_r J_s = J_s J_r = 0 \quad (1 \leq r \neq s \leq q). \quad (24)$$

Thus, in particular, for every s there is no non-zero idempotent K such that

$$J_s K = K J_s = K. \quad (25)$$

Finally, recall that for idempotents satisfying (24) (not necessary primitive ones) we have

$$\text{if } A_1, A_2, \dots, A_q \in \mathbf{D}(\mu) \text{ then } J_1 A_1 J_1 + J_2 A_2 J_2 + \dots + J_q A_q J_q \in \mathbf{D}(\mu); \quad (26)$$

for details cf. Jurek-Mason (1993), Theorem 2.3.6.

LEMMA 4. *There exists a positive integer q and a one parameter semigroup $\{C_w : w \in W\} \subset \mathbf{D}(\mu)$ (where W denotes the set of non-negative rational numbers) such that $\det C_w = e^{-qw}$ and $C_0 = I$.*

Proof. In view of Lemma 3, for the idempotents J_r in (24), let us choose $T_{n,r} \in \mathbf{D}(\mu)$ such that for $1 \leq r \leq q, n \geq 1$ we have

$$J_r T_{n,r} = T_{n,r} J_r = T_{n,r}, \quad T_{n,r} \rightarrow J_r, \quad \lim_{k \rightarrow \infty} T_{n,r}^k = 0, \quad 0 < \det_{J_r} T_{n,r} < 1. \quad (27)$$

Note that $\lim_{n \rightarrow \infty} (\log \det_{J_r} T_{n,r}) = 0$ and put $d(n, r) := [(-\log \det_{J_r} T_{n,r})^{-1}]$, where the bracket $[.]$ denotes the integer part. Hence, $\lim_{n \rightarrow \infty} d(n, r) = \infty$ and

$$\lim_{n \rightarrow \infty} (d(n, r) \cdot (-\log \det_{J_r} T_{n,r})) = 1, \quad (r = 1, 2, \dots, q). \quad (28)$$

Further, let W denote the set of all non-negative rational numbers (as in the statement of Lemma 4). Then

$$T_{n,r}^{[w d(n,r)]} \in \mathbf{D}(\mu) \quad \text{for all } n \in \mathbb{N}, w \in W, 1 \leq r \leq q; \quad \text{and by (26),}$$

$$\sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \in \mathbf{D}(\mu) \quad \text{for all } w \in W, \quad n \in \mathbb{N}; \quad (29)$$

Since $\mathbf{D}(\mu)$ is compact, there exist a subsequence $Q \subset \mathbb{N}$ and $C_w \in \mathbf{D}(\mu)$ such that for each $w \in W$

$$\sum_{r=1}^q T_{n,r}^{[w d(n,r)]} = \sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \rightarrow C_w, \text{ as } n \rightarrow \infty, n \in Q. \quad (30)$$

(Note for $w = 0$ that this gives $C_0 = I$ by (24).) Hence, from (28) and (30) we get

$$\det_{J_s} C_w = \lim_{n \rightarrow \infty, n \in Q} \det_{J_s} T_{n,s}^{[w d(n,s)]} = \lim_{n \rightarrow \infty, n \in Q} (\det_{J_s} T_{n,s})^{[w d(n,s)]} = e^{-w}. \quad (31)$$

So, by (17) and (24) we conclude

$$\begin{aligned} \det C_w &= \lim_{n \rightarrow \infty} \det_{J_1 + \dots + J_q} \left(\sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \right) \\ &= \prod_{r=1}^q \lim_{n \rightarrow \infty, n \in Q} \det_{J_r} T_{n,r}^{[w d(n,r)]} = e^{-qw} \end{aligned} \quad (32)$$

To show that $\{C_w : w \in W\}$ is indeed a one-parameter additive semigroup, note that for the integer part function $a \ni \mathbb{R} \rightarrow [a] \in \mathbb{Z}$ (integers) we have

$$[a + b] - [a] - [b] \in \{0, 1\} \quad (33)$$

(because $a + b - 1 < [a + b] \leq a + b$, $-a \leq -[a] < 1 - a$ and $-b \leq -[b] < 1 - b$)
Hence, for $w \in W$ and $u \in W$,

$$s_n := [(w + u) d(n, r)] - [w d(n, r)] - [u d(n, r)] \in \{0, 1\}, \quad (r = 1, 2, \dots, q)$$

Hence by (27)

$$\lim_{n \rightarrow \infty} J_r (T_{n,r}^{s_n} - I) J_r = 0, \quad (r = 1, 2, \dots, q), \quad (34)$$

since $s_n = 0$ or $s_n = 1$. Finally, from (30) and (24),

$$\begin{aligned} C_{w+u} - C_w C_u &= \\ \lim_{n \rightarrow \infty} \sum_{r=1}^q J_r T_{n,r}^{[(w+u) d(n,r)]} J_r - \left(\lim_{n \rightarrow \infty} \sum_{r=1}^q J_r T_{n,r}^{[w d(n,r)]} J_r \right) \left(\lim_{n \rightarrow \infty} \sum_{s=1}^q J_s T_{n,s}^{[u d(n,s)]} J_s \right) \\ &= \sum_{r=1}^q \lim_{n \rightarrow \infty} T_{n,r}^{[w d(n,r)] + [u d(n,r)]} J_r (T_{n,r}^{s_n} - I) J_r. \end{aligned}$$

Since $T_{n,r}^k \in \mathbf{D}(\mu)$ ($n, k \in \mathbb{N}, r = 1, 2, \dots, q$) and $\mathbf{D}(\mu)$ is compact (thus the norms of its members are bounded, say by B), one has from above and (34),

$$\|C_{w+u} - C_w C_u\| \leq B \sum_{r=1}^q \lim_{n \rightarrow \infty} \|J_r(T_{n,r}^{s_n} - I)J_r\| = 0,$$

which gives the one-parameter semigroup property $C_{w+u} = C_w C_u$.

LEMMA 5. *For the given (full) probability measure μ , its Urbanik decomposability semigroup $\mathbf{D}(\mu)$ contains at least one one-parameter semigroup $\{e^{-tQ}, t \geq 0\}$ (Q is a matrix) such that $e^{-tQ} \rightarrow 0$, as $t \rightarrow \infty$.*

Proof. Throughout this proof, we use freely all notations and arguments in the *proof* (as well as the statement) of Lemma 4.

Step 1. Let $\mathbf{S} := \overline{\{C_w : w \in W\}}$ (the closure in \mathbf{Aut}). Then \mathbf{S} is a compact semigroup in $\mathbf{D}(\mu)$. Further, since $\det C_w = e^{-qw}$, therefore it is an invertible operator. Thus

$\mathbf{H} := \{C_w : w \in W\} \cup \{C_w^{-1} : w \in W\}$ is a commutative group in \mathbf{Aut} .

To this end we have check that for $w, u \in W$, both $C_w C_u^{-1}$ and $C_w^{-1} C_u$ are in \mathbf{H} . Let assume that $w > u$ then $C_w C_u^{-1} = C_{w-u} C_u C_u^{-1} = C_{w-u} \in \mathbf{H}$. Similarly, $C_w^{-1} C_u = (C_u C_{w-u})^{-1} C_u = C_{w-u}^{-1} C_u^{-1} C_u = C_{w-u}^{-1} \in \mathbf{H}$. (These equations yield both closure and, with a trivial extra argument, commutativity.)

Step 2. Let $\mathbf{G} := \mathbf{S} \cup \mathbf{S}^{-1}$. Then $\mathbf{G} \subset \mathbf{Aut}$ is a commutative compactly generated subgroup. Moreover, the mapping $h : \mathbf{G} \rightarrow (\mathbb{R}, +)$ given by $h(A) := \log \det A$ is a homomorphism of those two topological groups with the kernel $\ker h = \mathbf{S}_0 := \mathbf{S} \cap \mathbf{A}(\mu)$. Thus the quotient group $\mathbf{G}/\ker h$ is isomorphic with $(\mathbb{R}, +)$.

To see the above claim, first of all note that since \mathbf{S}_0 is closed subsemigroup in the compact group $\mathbf{A}(\mu)$ therefore \mathbf{S}_0 is a compact group, by Theorem 1.1.12 in Paalman - de Miranda (1964) (see Theorem 2 in the Appendix).

If $A \in \mathbf{S}_0$ then $A \in \mathbf{A}(\mu)$ and by Corollaries 2.3.2 and 2.4.2 from Jurek and Mason (1993), we have that $|\det A| = 1$. On the other hand, since $A \in \mathbf{S}$ we have that $0 < \det A \leq 1$, so $\det A = 1$ and $h(A) = 0$ and $\mathbf{S}_0 \subset \ker h$.

Conversely, if $\det A = 1$ and $A \in \mathbf{S}$ then $A \in \mathbf{D}(\mu)$ and by Jurek-Mason (1993), Proposition 2.3.5 we get that $A \in \mathbf{A}(\mu)$. Consequently, $A \in \mathbf{S}_0$. If $\det A = 1$ and $A \in \mathbf{S}^{-1}$ then $A^{-1} \in \mathbf{S}$ and $\det A^{-1} = 1$ so $A \in \mathbf{S}_0$. This completes the proof of the Step 2.

Step 3. There is an isomorphism $g : \mathbf{G} \rightarrow \mathbb{R} \oplus \mathbf{S}_0$ between the two topological groups.

This is so, because \mathbf{G} is commutative and compactly generated group and the Pontriagin Theorem, from Montgomery and Zippin (1955), p. 187 (see Theorem 4 in the Appendix), gives the needed isomorphism.

Step 4. Taking the unit \mathcal{I} in the group in \mathbf{S}_0 and putting for $t \geq 0$,

$$\begin{aligned} T_t &:= g^{-1}(-t \oplus \mathcal{I}), & \text{if } g(\mathbf{S}) = (-\infty, 0] \oplus \mathbf{S}_0 \\ T_t &:= g^{-1}(t \oplus \mathcal{I}), & \text{if } g(\mathbf{S}) = [0, \infty) \oplus \mathbf{S}_0 \end{aligned} \quad (35)$$

we obtain the one-parameter semigroup of matrices in $\mathbf{D}(\mu)$.

>From the equality $g(\mathbf{G}) = g(\mathbf{S}) \cup (g(\mathbf{S}))^{-1}$, and the fact $g(\mathbf{S})$ is closed subsemigroup we infer that either $g(\mathbf{S}) = (-\infty, 0] \oplus \mathbf{S}_0$ or $g(\mathbf{S}) = [0, \infty) \oplus \mathbf{S}_0$.

Step 5. For $t \geq 0$, $T_t = \exp(-tV)$ for some matrix V , and $T_t \rightarrow 0$ as $t \rightarrow \infty$.

By Hille (1948), Theorem 8.4.2 (or Hille and Phillips (1957), Theorem 9.4.2 — see Theorem 3 in the Appendix — with the idempotent there being the identity matrix here in our context), we get the exponential form, that is, $T_t = \exp tQ, t \geq 0$, for some matrix Q .

For $t > 0$ we have that $T_t \notin \mathbf{S}_0$ and thus

$$0 < \det T_t < 1 \quad \text{for all } t > 0 \quad (36)$$

From the definitions of operators $C_w, T_{n,r}$ and semigroup \mathbf{S} it follows that the idempotents $J_r, 1 \leq r \leq q$ commute with \mathbf{S}

Since $T_t \in \mathbf{D}(\mu), t \geq 0$, the set $\{T_t, t \geq 0\}$ is conditionally compact. Hence by the Numakura Theorem, among the limits points (as $t \rightarrow \infty$) there is an idempotent, say K . Of course by (36) and a simple argument, $\det K = 0$; and by (24), $K = J_1 K + \dots + J_q K$. Also, K is the limit of a sequence of C_w 's with $w \rightarrow \infty$ (forced by (32) since $\det K = 0$), and hence by (31),

$$\det_{J_r} K = 0 \quad (r = 1, 2, \dots, q)$$

Since K and J_r commute and both are idempotents then so is $J_r K$. From above and (17), $\det_{J_r} J_r K = \det_{J_r} K = 0$, so $J_r \neq J_r K$. Moreover, we also have that $J_r(J_r K) = (J_r K)J_r = J_r K$. Thus from the properties of J_r ((24) and the *entire sentence* containing (25)) we must have $J_r K = 0$ and consequently $K = 0$. That is, the only limit point of T_t as $t \rightarrow \infty$. As a consequence, Lemma 5 holds.

Proof of Theorem 1. It follows from Lemma 5.

4. Appendix.

For an ease of reference let us quote here the following algebraic facts.

THEOREM 2. *Each locally compact subsemigroup S of a compact group G is a compact subgroup.*

Cf. A.B. Paalman - De Miranda (1964), Theorem 1.1.12.

THEOREM 3. *If $T : (0, \infty) \rightarrow \mathfrak{B}$ (a real or complex Banach algebra) satisfies*

$$T(t+s) = T(t)T(s) \text{ for all } 0 < t, s < \infty \text{ and } \lim_{t \rightarrow 0} T(t) = J \text{ (an idempotent),}$$

then there exists an element $A \in \mathfrak{B}$ such that

$$T(t) = J + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \quad (\text{absolutely convergent series}). \quad (37)$$

Cf. E. Hille (1948), Theorem 8.4.2 or E. Hille and R. Phillips (1957), Theorem 9.4.2.

THEOREM 4. (Pontriagin Theorem) *Suppose a topological group G' , generated by a compact set, contains a compact subgroup H' such that G'/H' is isomorphic with an r -dimensional real vector group V_r . Then G' has a vector subgroup E_r such that $G' = H' \oplus E_r$*

Cf. Montgomery and Zippin (1955), p. 187.

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